Dirac and Yang Monopoles Revisited

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Abstract

The Dirac monopoles in 3-space and their generalization by C. N. Yang to 5-space are observed to be just the Levi-Civita spin connections of the cylindrical Riemannian metric on the 3- and 5- dimensional punctured spaces respectively. Their straightforward generalization to higher dimensions is also investigated.

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1 Introduction

The main purpose of this note is to provide a geometric description of the Dirac and Yang monopoles from which a generalization to higher dimensions follows naturally. We believe that it is good to be aware of this observation for anyone who is interested in monopoles and their related objects in theoretical physics. For example, by using these generalized Dirac monopoles, the MICZ-Kepler problems has been extended [1] beyond dimension five.

In order to understand the background about our general result, it is helpful to have a quick review of the motivation and the main result of Ref. [2].

In Ref. [2], Yang starts with the following observation: the Dirac monopole with magnetic charge g is uniquely determined by the following two properties:

(a) For any closed surface Σ around the origin of $\mathbb{R}^3$, the magnetic flux through Σ is $4\pi g$ (g is a nonzero half integer), i.e., equivalently,

$$\int_\Sigma \frac{1}{2\pi} F = 2g. \tag{1}$$

(Here F is the field strength and is singular only at the origin of $\mathbb{R}^3$.)

(b) It is SO(3) symmetric.

Next he starts to search for a generalization of the Dirac monopole on $\mathbb{R}^3 \setminus \{0\}$ to an SU(2)-gauge field on $\mathbb{R}^5 \setminus \{0\}$, i.e., an SU(2)-gauge field on $\mathbb{R}^5 \setminus \{0\}$ satisfying

(a′) For any closed hyper-surface Σ around the origin of $\mathbb{R}^5$,

$$\int_\Sigma \mathrm{Tr} F^2 \neq 0, \tag{2}$$

(Here F is the field strength and is singular only at the origin of $\mathbb{R}^5$; and Tr is the trace in the defining representation of SU(2).)

(b′) It is SO(5) symmetric.

Here is Yang's main result in Ref. [2]: There are two and only two solutions α and β satisfying (a′) and (b′) and are characterized by

$$\int_\Sigma \frac{1}{8\pi^2} \mathrm{Tr} F^2 = 1 \quad \text{and} \quad \int_\Sigma \frac{1}{8\pi^2} \mathrm{Tr} F^2 = -1 \tag{3}$$

respectively.

In Ref. [2], the local formula for vector potentials of the monopoles are written down explicitly, but no conceptual reason is given. However, the two characterization properties of Yang monopoles force us to believe that there must be a clean conceptual description of the monopoles in terms of geometry, and that belief turns out to be true.

1.1 Main Results

In terms of spherical coordinates, the flat metric on $\mathbb{R}^{2n+1}$ is

$$ds_E^2 = dr^2 + r^2 d\Omega^2 \tag{4}$$

where r is the radius and $d\Omega^2$ is the round metric for the unit sphere. On $\mathbb{R}^{2n+1} \setminus \{0\}$, the flat metric is conformally equivalent to the **cylindrical metric**:

$$ds^2 = \frac{1}{r^2} ds_E^2 = \frac{1}{r^2} dr^2 + d\Omega^2. \tag{5}$$

Note that SO$(2n+1)$ is clearly a group of isometry on the Riemannian manifold $(\mathbb{R}^{2n+1} \setminus \{0\}, ds^2)$. Note also that the gamma matrix associated to the unit radial directional vector of $\mathbb{R}^{2n+1} \setminus \{0\}$ splits the spin bundle on $(\mathbb{R}^{2n+1} \setminus \{0\}, ds^2)$ into two sub vector bundles of rank 2^{n-1}, with respect to this splitting, the Levi-Civita spin connection splits into A_+ and A_- on the respective sub vector bundles. Note also that Spin$(2n+1)$ is a group of symmetry of the plus (or minus) spin bundle which leaves A_+ (or A_-) invariant.

We are now ready to state

Theorem 1.1 (Main Theorem)**.** *Let (A_+, A_-) be the Levi-Civita spin connection on the Riemannian manifold $(\mathbb{R}^{2n+1} \setminus \{0\}, ds^2)$, then*

1. *$A_\pm$ are $\mathrm{U}(2^{n-1})$-gauge fields on $\mathbb{R}^{2n+1} \setminus \{0\}$, in fact $\mathrm{SU}(2^{n-1})$-gauge fields on $\mathbb{R}^{2n+1} \setminus \{0\}$ if $n > 1$.*

2. *Let $F_\pm$ be the gauge field strength respectively, Σ a closed hyper-surface around the origin of $\mathbb{R}^{2n+1}$, then*

$$\int_\Sigma \frac{1}{n!}\mathrm{Tr}(-\frac{F_\pm}{2\pi})^n = \pm 1 \tag{6}$$

respectively, where Tr *is the trace in the defining representation of* $\mathrm{U}(2^{n-1})$.

3. *When $n=1$, $A_\pm$ are respectively the Dirac monopole with fundamental plus or minus magnetic charge $g=\pm\frac{1}{2}$.*

4. *When $n=2$, $A_\pm$ are respectively the Yang monopoles α and β introduced in Ref. [2].*

Remark 1. *One can show that, when $n>1$, there are no other* $\mathrm{SU}(2^{n-1})$*-gauge field on $(\mathbb{R}^{2n+1}\setminus\{0\}, ds^2)$ such that 1) it is* $\mathrm{Spin}(2n+1)$*-symmetric, 2) $\int_\Sigma \mathrm{Tr}(-\frac{F_\pm}{2\pi})^n$ is nontrivial for the field strength F and closed hyper-surface Σ around the origin of $\mathbb{R}^{2n+1}$. But this is less interesting, so its proof will be omitted.*

Added in May, 2007. The results presented here were posted in the archive as math-ph/0409051 in September, 2004. I was told by I. Cotaescu that he wrote Ref. [3] in the winter 2004/2005 in which virtually the same results were obtained. While we had different motivations, we reached the same destination independently.

2 Proof of Theorem 1.1

Here we assume the readers are familiar with the basic facts on Clifford algebra and spin groups, a good reference is Ref. [4]. We assume $e_1, \ldots, e_k$ are the standard basis of $\mathbb{R}^k$ and if x, y are in $\mathbb{R}^k$ then xy is the multiplication of x with y in Clifford algebra. Note that $xy+yx=-2x\cdot y$ where $x\cdot y$ is the dot product of x with y. We also assume the reader is familiar with the definition of connection on a principal bundle in the sense of Ehresmann (see page 358 of Ref. [5]).

In our recent paper [6], we observe that there is a canonical principal $\mathrm{Spin}(2n)$-bundle over S^{2n}:

$$\mathrm{Spin}(2n)\to\mathrm{Spin}(2n+1)\to\mathrm{S}^{2n} \tag{7}$$

with a canonical $\mathrm{Spin}(2n+1)$-symmetric $\mathrm{Spin}(2n)$-connection

$$\omega(g)=\mathrm{Pr}(g^{-1}dg) \tag{8}$$

where Pr is the orthogonal projection onto the Lie algebra of $\mathrm{Spin}(2n)$. It is clear that $\omega(hg)=\omega(g)$ for any $h\in\mathrm{Spin}(2n+1)$. Note that the bundle and the connection here are precisely the principle spin bundle and the Levi-Civita connection of S^{2n} equipped with the standard round metric.

As a Riemannian manifold, $(\mathbb{R}^{2n+1}\setminus\{0\}, ds^2) = \mathbb{R}\times \mathrm{S}^{2n}$ where $\mathbb{R}$ is standard real line with the standard flat metric and S^{2n} is the standard unit sphere with the standard round metric. Let

$$p_2: \ \mathbb{R}\times \mathrm{S}^{2n} \to \mathrm{S}^{2n}$$

be the projection map, then it is clear that the spin bundle on $(\mathbb{R}^{2n+1}\setminus\{0\}, ds^2)$ is just the pullback by p_2 of the spin bundle on S^{2n}, so is the spin connection.

Note that $\mathrm{Spin}(2n)$ has two fundamental spin representations $\rho_\pm$ of dimension 2^{n-1}, so we have two associated vector bundles over $\mathbb{R}^{2n+1}\setminus\{0\}$ of rank 2^{n-1}, hence two associated canonical $\mathrm{Spin}(2n+1)$-symmetric $\mathrm{U}(2^{n-1})$-connections $A_\pm$ — the ones we mentioned in the introduction.

We are ready to present the proof of our main theorem.

Step1 : To get explicit expressions for the canonical $\mathrm{Spin}(2n+1)$-symmetric connection $\omega(g) = \mathrm{Pr}(g^{-1}dg)$, we need to choose an open cover (a family of open sets of $\mathbb{R}^{2n+1}\setminus\{0\}$ whose union is equal to $\mathbb{R}^{2n+1}\setminus\{0\}$) and a gauge over each of the open sets of the open cover, i.e., a smooth section of the principal bundle over each of the open sets of the open cover.

The open cover chosen here consists of two open sets: $V_\pm$, where $V_\pm$ is $\mathbb{R}^{2n+1}$ with the positive/negative $(2n+1)$-st axis removed, i.e., $\mathbb{R}\times \mathrm{S}_\pm$ where $\mathrm{S}_\pm$ is the sphere with north/south pole removed.

A point on V_- can be written as $x = r\cos\theta e_{2n+1} + r\sin\theta y$ where $0\le\theta<\pi$, $y\in \mathrm{S}^{2n-1}$ and $r>0$. On V_-, we choose this gauge:

$$g(x) = \cos\frac{\theta}{2} - \sin\frac{\theta}{2} y e_{2n+1}. \tag{9}$$

(it is easy to see that $x = g(x)e_{2n+1}g(x)^{-1}$, so g is indeed a smooth section over V_-.) Then, under this choice of gauge, we can calculate the gauge potential $\mathscr{A}(x) := -i\omega(g(x)) = -i\mathrm{Pr}(g^{-1}(x)dg(x))$ and get

$$\boxed{\mathscr{A}(x) = i(\sin\tfrac{\theta}{2})^2 y\, dy}\,; \tag{10}$$

we can also calculate the gauge field strength $\mathscr{F} = d\mathscr{A} + i\mathscr{A}^2$ and get

$$\mathscr{F}(x) = \frac{i}{2}\sin\theta\, d\theta\, y\, dy + \frac{i}{4}(\sin\theta)^2\, dy\, dy. \tag{11}$$

A point on V_+ can be written as $x = r\cos\theta e_{2n+1} + r\sin\theta y$ where $0<\theta\le\pi$, $y\in \mathrm{S}^{2n-1}$ and $r>0$. On V_+, we choose this gauge:

$$\tilde{g}(x) = \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2} y e_{2n+1}\right) y e_{2n} = \cos\frac{\theta}{2} y e_{2n} - \sin\frac{\theta}{2} e_{2n+1}e_{2n}. \tag{12}$$

Under this choice of gauge, we have

$$\boxed{\tilde{\mathscr{A}}(x) = -i(\cos\tfrac{\theta}{2})^2 e_{2n} y\, dy e_{2n}} \tag{13}$$

and

$$\tilde{\mathscr{F}}(x) = e_{2n} y\, \mathscr{F}(x)\, y e_{2n} = \frac{i}{2}\sin\theta\, d\theta\, e_{2n} y\, dy e_{2n} - \frac{i}{4}(\sin\theta)^2\, e_{2n} dy\, dy e_{2n}. \quad (14)$$

Let $A_\pm = \rho_\pm(\mathscr{A})$ and $F_\pm = \rho_\pm(\mathscr{F})$. The only nontrivial topological number we are calculating here is

$$c_\pm := \int_{\mathrm{S}^{2n}} \frac{1}{n!}\mathrm{Tr}\left(-\frac{F_\pm}{2\pi}\right)^n = \int_{\mathrm{S}^{2n}} \frac{1}{n!}\mathrm{Tr}_{\rho_\pm}\left(-\frac{\mathscr{F}}{2\pi}\right)^n = \int_{\mathrm{S}^{2n}} \frac{1}{n!}\mathrm{Tr}_{\rho_\pm}\left(-\frac{\tilde{\mathscr{F}}}{2\pi}\right)^n, \quad (15)$$

where S^{2n} should be viewed as the hyper-surface $1\times \mathrm{S}^{2n}$ in $\mathbb{R}\times\mathrm{S}^{2n}$.

And the calculation of $c_\pm$ is not hard:

$$\begin{aligned} c_\pm &= \frac{2}{(n-1)!}\left(-\frac{i}{8\pi}\right)^n \int_{\mathrm{S}^{2n}} (\sin\theta)^{2n-1}\, d\theta \mathrm{Tr}_{\rho_\pm}\left(y\, dy (dy\, dy)^{n-1}\right) \\ &= \frac{2}{(n-1)!}\left(-\frac{i}{8\pi}\right)^n \int_0^\pi (\sin\theta)^{2n-1}\, d\theta \int_{\mathrm{S}^{2n-1}} \mathrm{Tr}_{\rho_\pm}\left(y\, dy (dy\, dy)^{n-1}\right) \\ &= \pm \frac{n(2n-1)!!}{(2\pi)^n}\mathrm{vol}(\mathrm{B}^{2n}) \int_0^\pi (\sin\theta)^{2n-1}\, d\theta \\ &= \pm 1. \end{aligned} \quad (16)$$

Here $\mathrm{vol}(\mathrm{B}^{2n})$ is the volume of the unit $2n$-ball and we assume that the orientation on spheres is the standard one and the convention that

$$\rho_\pm(e_1\cdots e_{2n}) = \pm i^n I_{2^{n-1}} \quad (17)$$

is adopted, here $I_{2^{n-1}}$ is the identity matrix of order 2^{n-1}.

In the above calculation, if we replace S^{2n} by any closed hyper-surface Σ around the origin of $\mathbb{R}^{2n+1}$, the result is still the same, that is because of the Stokes theorem and the Bianchi identity. So point 2 of the main theorem is proved.

Step2 : Note that (10) and (13) are the explicit formulae for the gauge potentials, and (11) and (14) are the explicit formulae for the corresponding gauge field strengths. When $n>1$, we claim that $\rho_\pm(\mathscr{A})$ and $\rho_\pm(\tilde{\mathscr{A}})$ are $\mathrm{SU}(2^{n-1})$-gauge potentials, that is because $\mathrm{Tr}_{\rho_\pm}(e_i e_j) = 0$ if $i\neq j$: For example,

$$\begin{aligned} \mathrm{Tr}_{\rho_\pm}(e_1e_2) &= \mathrm{Tr}_\rho\left(e_1e_2\frac{1}{2}(1\pm(-i)^n e_1\cdots e_{2n})\right) \\ &= \frac{1}{2}\left(\mathrm{Tr}_\rho(e_1e_2) \pm (-i)^n \mathrm{Tr}_\rho(e_3e_4\cdots e_{2n})\right) \\ &= \frac{1}{2}\left(\mathrm{Tr}_\rho(e_2e_1) \pm (-i)^n \mathrm{Tr}_\rho(e_{2n}e_3e_4\cdots e_{2n-1})\right) \\ &= -\frac{1}{2}\left(\mathrm{Tr}_\rho(e_1e_2) \pm (-1)^{n-1} \mathrm{Tr}_\rho(e_3e_4\cdots e_{2n})\right) \\ &= 0. \end{aligned} \quad (18)$$

So point 1 of the main theorem is proved.

Step3 : To prove points 3 and 4, we can do the explicit calculations and then compare with explicit formulae in Ref. [2]. A quick way to prove them is to use the two characterization properties of the Dirac or Yang monopoles that we mentioned in the introduction.

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